Algorithmic Inversion Principles in Extensional Type Theory

Andrej Bauer *Philipp G. Haselwarter*

University of Ljubljana, Slovenia

EUTypes Nijmegen 2018

Inversion principles

Input: a derivable judgement

$$\Gamma \vdash (\lambda x{:}A.B \,.\, s) : C$$

Inversion principles

Input: a derivable judgement

$$\Gamma \vdash (\lambda x{:}A.B\,.\,s) : C$$

Question: Is there a derivation starting with a particular shape?

$$\dfrac{\dfrac{\mathcal{D}_1}{\Gamma,\, x : A \vdash s : B}}{\Gamma \vdash (\lambda x{:}A.B\,.\,s) : \prod_{(x:A)} B} \qquad \dfrac{\mathcal{D}_2}{\Gamma \vdash \prod_{(x:A)} B \equiv C}$$
$$\overline{\qquad\Gamma \vdash (\lambda x{:}A.B\,.\,s) : C\qquad}$$

Inversion principles

Input: a derivable judgement

$$\Gamma \vdash (\lambda x{:}A.B \,.\, s) : C$$

Question: Is there a derivation starting with a particular shape?

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\Gamma,\, x : A \vdash s : B}}{\Gamma \vdash (\lambda x{:}A.B \,.\, s) : \prod_{(x:A)} B} \qquad \dfrac{\mathcal{D}_2}{\Gamma \vdash \prod_{(x:A)} B \equiv C}}{\Gamma \vdash (\lambda x{:}A.B \,.\, s) : C}$$

Answer: Yes! Also for Extensional Type Theory (ETT).

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.

$$\frac{\Gamma \vdash A \ \ \mathsf{type} \qquad \Gamma, x : A \vdash B \ \ \mathsf{type}}{\Gamma \vdash \prod_{(x:A)} B \ \ \mathsf{type}}$$

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.

$$\frac{\Delta \vdash A \;\; \mathsf{type} \qquad \Xi, x : A \vdash B \;\; \mathsf{type}}{\Gamma \vdash \prod_{(x:A)} B \;\; \mathsf{type}}$$

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.
Contexts in premises must be independent!

$$\frac{\Delta \nvdash^{f} A \ \text{type} \qquad \Xi, x : A \nvdash^{f} B \ \text{type} \qquad \Delta \leq \Gamma \qquad \Xi \leq \Gamma}{\Gamma \nvdash^{f} \prod_{(x:A)} B \ \text{type}}$$

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.
Contexts in premises must be independent!

$$\frac{\Delta \Vdash A \text{ type} \qquad \Xi, x : A \Vdash B \text{ type} \qquad \Delta \leq \Gamma \qquad \Xi \leq \Gamma}{\Gamma \Vdash \prod_{(x:A)} B \text{ type}}$$

New side-conditions: $\Delta \leq \Gamma$

- at least: $\forall \Gamma,\ \bullet \leq \Gamma$ and $\Gamma \leq (\Gamma, x : A)$
- at most: if $\Delta \leq \Gamma$ and $\Delta \Vdash \mathcal{J}$, then $\Gamma \Vdash \mathcal{J}$

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.
Contexts in premises must be independent!

$$\frac{\Delta \Vdash^{f} A \ \text{type} \qquad \Xi, x : A \Vdash^{f} B \ \text{type} \qquad \Delta \leq \Gamma \qquad \Xi \leq \Gamma}{\Gamma \Vdash^{f} \prod_{(x : A)} B \ \text{type}}$$

New side-conditions: $\Delta \leq \Gamma$

- at least: $\forall \Gamma$, $\bullet \leq \Gamma$ and $\Gamma \leq (\Gamma, x : A)$
- at most: if $\Delta \leq \Gamma$ and $\Delta \Vdash^{f} \mathcal{J}$, then $\Gamma \Vdash^{f} \mathcal{J}$

Example: $x : A, y : B \leq x : \bot, y : C$

Sub-contexts and compatible contexts

For forward reasoning we need to *combine* judgements.
Contexts in premises must be independent!

$$\frac{\Delta \vDash^{\!f} A \ \text{type} \qquad \Xi, x : A \vDash^{\!f} B \ \text{type} \qquad \Delta \leq \Gamma \qquad \Xi \leq \Gamma}{\Gamma \vDash^{\!f} \prod_{(x:A)} B \ \text{type}}$$

New side-conditions: $\Delta \leq \Gamma$

- at least: $\forall \Gamma$, $\bullet \leq \Gamma$ and $\Gamma \leq (\Gamma, x : A)$
- at most: if $\Delta \leq \Gamma$ and $\Delta \vDash^{\!f} \mathcal{J}$, then $\Gamma \vDash^{\!f} \mathcal{J}$

Example: $x : A,\ y : B \leq x : \bot,\ y : C$
Non-example: $x : A,\ y : B \not\leq x : A,\ z : B$

Axioms for compatible contexts

$$\frac{\Delta \Vdash^{f} A \ \text{type} \qquad \Xi, x : A \Vdash^{f} B \ \text{type} \qquad \{\Delta, \Xi\} \uparrow \Gamma}{\Gamma \Vdash^{f} \prod_{(x:A)} B \ \text{type}}$$

The relation $\{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma$ generalises sub-contexts.

- it must satisfy:

$$\{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma \implies \forall j \leq n\,,\ \Gamma_j \leq \Gamma$$
$$(\forall j \leq n\,,\ \Gamma_j \leq \Gamma) \implies \exists \Gamma' \leq \Gamma\,,\ \{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma'$$

- we can choose to compute Γ from $\Gamma_1, ..., \Gamma_n$

Axioms for compatible contexts

$$\frac{\Delta \nvdash^{f} A \ \text{type} \qquad \Xi' \nvdash^{f} B \ \text{type} \qquad \Xi' \backslash (x{:}A) \sim \Xi \qquad \{\Delta, \Xi\} \uparrow \Gamma}{\Gamma \nvdash^{f} \prod_{(x:A)} B \ \text{type}}$$

The relation $\{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma$ generalises sub-contexts.

- it must satisfy:

$$\{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma \implies \forall j \leq n\,,\ \Gamma_j \leq \Gamma$$
$$(\forall j \leq n\,,\ \Gamma_j \leq \Gamma) \implies \exists \Gamma' \leq \Gamma\,,\ \{\Gamma_1, ..., \Gamma_n\} \uparrow \Gamma'$$

- we can choose to compute Γ from $\Gamma_1, ..., \Gamma_n$

The relation $\Xi' \backslash (x{:}A) \sim \Xi$ generalises $\Xi' = \Xi,\ x : A$.
Don't worry about this.

Relating the new theory to ETT

Proposition (Conservativity)

If $\Gamma \Vdash^{f} \mathcal{J}$ *then* $\Gamma \vdash \mathcal{J}$*.*

Proposition (Completeness)

If $\Gamma \vdash \mathcal{J}$ *then there is* $\Delta \leq \Gamma$ *such that* $\Delta \Vdash^{f} \mathcal{J}$*.*

Strengthening

If $\Gamma, x : A \vdash s : B$ derivable and $x \notin \mathit{FreeVar}(s, B)$,
then $\Gamma \vdash s : B$?

Strengthening

If $\Gamma, x : A \vdash s : B$ derivable and $x \notin \mathit{FreeVar}(s, B)$,
then $\Gamma \vdash s : B$?

Answer: ITT: yes, ETT: no.

Strengthening

If $\Gamma, x : A \vdash s : B$ derivable and $x \notin \textit{FreeVar}(s, B)$,
then $\Gamma \vdash s : B$?

Answer: ITT: yes, ETT: no.

Example:

$$\dfrac{x : A,\, p : A = B \vdash x : A \qquad \dfrac{x : A,\, p : A = B \vdash p : A = B}{x : A,\, p : A = B \vdash A \equiv B}}{x : A,\, p : A = B \vdash x : B}$$

Strengthening and inversion

Example: Deconstruct an implication

$$\Gamma \nvDash A \to B \ \text{ type}$$

Really, this is

$$\Gamma \nvDash \prod_{(\_ : A)} B \ \text{ type}$$

Inversion only yields $\Gamma, x : A \nvDash B$ type, but B is independent of x!

Idea: annotated type theory

Assumption sets

$$\alpha, \beta, \gamma \ ::= \quad \{x_1, ..., x_n\}$$

Judgements

$$\Gamma \vdash^{\alpha} A \ \text{type} \ \mid \ \Gamma \vdash^{\gamma} s : A \ \mid \ \ldots$$

Contexts

$$\Gamma, \Delta \ ::= \quad \bullet \ \mid \ \Gamma, x : A^{\alpha}$$

Types

$$A, B \ ::= \quad \prod_{(x : A^{\alpha})} B^{\beta} \ \mid \ \mathsf{Eq}_{A^{\alpha}}(s^{\sigma}, t^{\tau}) \ \mid \ \ldots$$

Recovering strengthening

Proposition

Given $\Gamma \vdash^{\gamma} s : A$*, there exists a context* $\Gamma_{|\gamma}$*, such that* $\gamma = \mathrm{dom}(\Gamma_{|\gamma})$*,* $\Gamma_{|\gamma} \leq \Gamma$*, and* $\Gamma_{|\gamma} \vdash^{\gamma} s : A$*.*

Annotated TT: Product formation

$$\frac{\Delta \vdash^{\alpha} A \ \textsf{type} \qquad \Xi \vdash^{\beta} B \ \textsf{type} \qquad \Xi \backslash (x{:}A^{\alpha}) \sim \Xi' \qquad \{\Delta, \Xi'\} \uparrow \Gamma \qquad \alpha \cup (\beta \backslash \{x\}) \subseteq \gamma \qquad \gamma \subseteq \mathrm{dom}(\Gamma)}{\Gamma \vdash^{\gamma} \prod_{(x:A^{\alpha})} B^{\beta} \ \textsf{type}}$$

Annotated TT: Conversion

$$\frac{\Xi \vdash^{\xi} s : A \qquad \Delta \vdash^{\delta} A \equiv B \qquad \{\Xi, \Delta\} \uparrow \Gamma}{\Gamma \vdash^{\gamma} s : B}$$

Annotated TT: Conversion

$$\frac{\Xi \vdash^{\xi} s : A \qquad \Delta \vdash^{\delta} A \equiv B \qquad \{\Xi, \Delta\} \uparrow \Gamma \qquad \xi \cup \delta \subseteq \gamma \qquad \gamma \subseteq \mathrm{dom}(\Gamma)}{\Gamma \vdash^{\gamma} s : B}$$

Relating ETT and annotated TT

Definition. Given an annotated context Γ (resp. judgement $\mathscr{J}$), its stripping $\underline{\Gamma}$ (resp. $\underline{\mathscr{J}}$) is given by deleting all annotations.

Relating ETT and annotated TT

Definition. Given an annotated context Γ (resp. judgement $\mathcal{J}$), its stripping $\underline{\Gamma}$ (resp. $\underline{\mathcal{J}}$) is given by deleting all annotations.

Proposition (Conservativity of ATT)

If $\Gamma \vdash^{\gamma} \mathcal{J}$ then there exists $\underline{\mathcal{J}}$, such that $\Gamma \vdash \underline{\mathcal{J}}$.

Relating ETT and annotated TT

Definition. Given an annotated context Γ (resp. judgement $\mathcal{J}$), its stripping $\underline{\Gamma}$ (resp. $\underline{\mathcal{J}}$) is given by deleting all annotations.

Proposition (Conservativity of ATT)

If $\Gamma \vdash^{\gamma} \mathcal{J}$ *then there exists* $\underline{\mathcal{J}}$, *such that* $\Gamma \vdash \underline{\mathcal{J}}$.

Proposition (Completeness of ATT)

If $\Gamma \vdash \mathcal{J}$ *then there are* Γ', γ, *and* $\mathcal{J}'$, *such that* $\underline{\Gamma}' \leq \Gamma$, $\underline{\mathcal{J}}' = \mathcal{J}$ *and* $\Gamma' \vdash^{\gamma} \mathcal{J}'$.

Relating ETT and annotated TT

Definition. Given an annotated context Γ (resp. judgement $\mathcal{J}$), its stripping $\underline{\Gamma}$ (resp. $\underline{\mathcal{J}}$) is given by deleting all annotations.

Proposition (Conservativity of ATT)

If $\Gamma \vdash^{\gamma} \mathcal{J}$ *then there exists* $\underline{\mathcal{J}}$*, such that* $\Gamma \vdash \underline{\mathcal{J}}$*.*

Proposition (Completeness of ATT)

If $\Gamma \vdash \mathcal{J}$ *then there are* Γ'*,* γ*, and* $\mathcal{J}'$*, such that* $\underline{\Gamma}' \leq \Gamma$*,* $\underline{\mathcal{J}}' = \mathcal{J}$ *and* $\Gamma' \vdash^{\gamma} \mathcal{J}'$*.*

Thank you

Abstraction relation

We assume given a relation $\Delta\backslash(x{:}A) \sim \Delta'$ which satisfies the following conditions:

$$\Delta\backslash(x{:}A) \sim \Delta' \implies \exists \Phi \leq \Delta'\,,\ \Delta \leq (\Phi,\, x : A) \qquad \text{(abs-elim)}$$

$$\Delta \leq (\Gamma,\, x : A) \implies \exists \Delta' \leq \Gamma\,,\ \Delta\backslash(x{:}A) \sim \Delta' \qquad \text{(abs-intro)}$$